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METHODS OF CONSTRUCTING OPTIMAL STABILIZERS†

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The problem of transforming a linear dynamical system in the neighbourhood of a state of equilibrium [1, 2] is solved using the special problem of the damping of the system by controls of minimum intensity after a finite time interval. The possibility of using other problems of optimal control is discussed. The main attention is devoted to constructing algorithms of the operation of a device (a stabilizer) which is able, in real time, to generate a stabilizing control circulating in the closed optimal system when unknown perturbations operate constantly [3, 4]. The proposed method is based on the constructive theory of optimal control [5, 6]. Another form of this theory for solving the problem of stabilization is presented in [7] (see also [8]).

1. STATEMENT OF THE PROBLEM

LET THE behaviour of a dynamical system be described by the equation

$$\dot{x} = Ax, \quad x \in R^n \tag{1.1}$$

We will assume that the behaviour of the dynamical system under the action of a control is described by the equation

$$\dot{x} = Ax + bu, \quad t \geqslant 0 \tag{1.2}$$

The classic statement of the problem of stabilization is given in [1]. Let us examine this problem from a somewhat different standpoint. We specify the finite time interval of programmed damping θ, $0<\theta<+\infty$, and the accuracy of damping $\varepsilon>0$. We shall control the dynamical system (1.2) by means of piecewise-continuous functions $u(t)$, $t \geqslant 0$.

Definition 1. The dynamical system (1.2) is called dampened from the state x_0 if a permissible control $u(\cdot)=(u(t),\ t\geqslant 0)$ exists which generates a trajectory $x(t)=x(t,\ x_0,\ u(\cdot))$ of the system such that the conditions

$$|x_i(\theta)| \leqslant \varepsilon, \quad i \in I = \{1, 2, \ldots, n\} \tag{1.3}$$

are satisfied.

Definition 2. The system is called dampened if property (1.3) is satisfied for each initial state $x_0 \in \mathbf{R}^n$ of the system (1.2).

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It is obvious that for system (1.1) to be dampened for all sufficiently small $\varepsilon > 0$, it is necessary and sufficient that system (1.2) is completely controllable in the Kalman sense.

We will estimate the performance of the damping control $u(t)$, $t \geq 0$ by the magnitude of the functional

$$\mu(u) = \max_{0 \leq t \leq \theta} |u(t)| \tag{1.4}$$

Definition 3. The damping control $u^0(t)$, $t \geq 0$ is said to be program-optimal if the performance criterion (1.4) has the minimum value

$$\mu(u^0) = \min \mu(u) \tag{1.5}$$

for it.

The above concepts are not beyond the scope of the classical theory of controllability. To construct the program-optimal control a method based on the solution of the problem of moments is known [2]. The method developed in [3] for solving problem (1.2)–(1.5) is used below.

The construction of program-optimal controls is only one aspect (and not the most important one for applications) of the problem of the optimal stabilization of dynamical systems. For the practical stabilization of control systems the problem of constructing feedback-type positional optimal controls is considerably more difficult and important. The aim of our further discussion is to describe an algorithm of the operation of a device which solves the problem of optimal stabilization when generating feedback-type optimal controls in problem (1.2)–(1.5). We will first introduce some necessary definitions.

We imbed problem (1.2)–(1.5) in the family of problems

$$\begin{gathered} \max_{0 \leq t \leq \theta} |u(t)| \to \min \\ \dot{x} = Ax + bu, \quad x(0) = z; \quad |x_i(\theta)| \leq \varepsilon, \quad i \in I \end{gathered} \tag{1.6}$$

which depend on the n-vector z. We denote the program solution of (1.6) by $u^0(t \mid z)$, $x^0(t \mid z)$, $t \in [0, \theta]$.

Definition 4. The family of piecewise-continuous functions $(u^0(t \mid z),\ t \in T_\theta = [0, \theta],\ z \in R)$ is called feedback-type optimal control in problem (1.2)–(1.5).

We will call the function $u(t \mid x(t_k))$, $t_k \leq t \leq t_{k+1}$, $t_k = k\theta$, $k = 0, 1, \ldots$, the program-positional control.

As we know, it is usual to call the function $u(t \mid x(t_k))$, $t \in T$ the positional control.

We mean by the motion of the closed system the solution $x(t)$, $t \geq 0$, of the equation

$$\begin{gathered} \dot{x} = Ax + bu^0(t \mid x(t_k)), \quad t_k \leq t \leq t_{k+1} \\ t_k = k\theta, \quad x(0) = x_0, \quad k = 0, 1, 2, \ldots \end{gathered} \tag{1.7}$$

By the definition of the optimal feedback, the closed system (1.7) is asymptotically stable

$$\max_{i \in I} |x_i(t)| \to 0 \qquad \text{as} \qquad t \to \infty$$

Equation (1.7) describes the behaviour of the dynamical system closed by feedback under ideal conditions. In this case the problem of optimal stabilization is reduced to solving a denumerable set of the program problems (1.6). The last problems may be solved in succession by preparing the solution in each current interval $T_k = [t_k, t_{k+1}]$ for the next interval T_{k+1}. If powerful computers are available and sufficiently large magnitudes of θ this approach can be realized. In practice, however, the closed system will constantly be subjected to the action of unknown perturbations.

Suppose that its actual motion is described by the equation

$$\dot{x} = Ax + bu^0(t \mid x(t_k)) + w(t), \quad x(0) = x_0$$
$$t \in T_k, \quad k = 0,1,2,\ldots \tag{1.8}$$

where $w(t)$, $t \geqslant 0$, is an unknown piecewise-continuous function.

Let the perturbation $w^*(t)$, $t \geqslant 0$, be realized in a certain particular process of operation of the dynamical system. It will generate the trajectory $x^*(t)$, $t \geqslant 0$, of Eq. (1.8). The control which will then circulate in the closed system has the form

$$u^*(t) = u^0(t \mid x^*(t_k)), \quad t \in T_k, \quad k = 0,1,2,\ldots$$

Now it is impossible to know in advance which of the states $x^*(t_k)$, $k = 1, 2, \ldots$, are realized, and hence, in the context of the approach mentioned, it is necessary to prepare the family $(u^0(t \mid z),\ t \in T_\theta)$ for all possible vectors $z \in \mathbb{R}^n$ in advance. It is obvious that this problem is not simpler than the classical problem of synthesizing the optimal feedback, which has not yet been solved.

Definition 5. A device which is able to generate the control $u^*(t)$, $t \geqslant 0$, in real time for each specific process of operation of the dynamical system is called the optimal stabilizer (OS) of the dynamical system (1.1).

The algorithm of the operation of the OS will be described in Section 3. For the moment we will consider some necessary additional information.

2. THE GOVERNING EQUATIONS OF THE OPTIMAL STABILIZER

Consider the problem

$$\alpha(z,\mu) = \min_u \max_{1 \leqslant i \leqslant n} |x_i(\theta)|$$
$$\dot{x} = Ax + bu, \quad x(0) = z; \quad |u(t)| \leqslant \mu, \quad t \in T_\theta \tag{2.1}$$

together with problem (1.6).

It is obvious that the minimum number $\mu = \mu(z)$ satisfying the equation $\alpha(z, \mu(z)) = \varepsilon$ is the optimal magnitude of the performance criterion of problem (1.7). The criterion of optimality for problem (2.1) can be obtained following [3]. By this criterion the optimal control $u_\mu^0(t) = u_\mu^0(t \mid z)$, $t \in T_\theta$ has the form

$$u_\mu^0(t) = \mu(z)\operatorname{sign}\Delta_\mu^0(t)$$
$$\Delta_\mu(t) = \Delta_\mu(t \mid z) = \psi_\mu'(t)b = -y'(z)F(\theta - t)b$$
$$y(z) \in R^n, \quad y_i(z) = 0, \quad i \in I_H(z), \quad I_H(z) = I \setminus I(z)$$
$$I(z) = \{i \in I: |x_i(\theta)| = \alpha(z,\mu)\}, \quad \dot{F} = AF, \quad F(0) = E$$

where $\Delta_\mu(t)$ is the optimal vector of potentials.

Hence the optimal control $u_\mu^0(t)$, $t \in T$, is specified by the quantity $\mu(z)$, the set $I(z)$ and

$$t_j(z), \quad j \in P = \{1,\ldots,p\}; \quad y(z) \tag{2.2}$$

which consists of zeros $t_1(z) < \dots < t_p(z)$, the cocontrol $\Delta_\mu(t)$, $t \in T$ and the vector of potentials.
The elements (2.2) satisfy the system of equations

$$\begin{aligned} &f_k(t_j, j \in P; z) = 0, \quad k \in I(z) \\ &q_l(t_j, j \in P; y) = 0, \quad l \in P \\ &\sum_{x_i(\theta)=\alpha(z,\mu),\ i \in I(z)} y_i - \sum_{x_i(\theta)=-\alpha(z,\mu),\ i\in I(z)} y_i = 1 \end{aligned} \tag{2.3}$$

Here

$$f_k(t_j, j \in P; z) = \sum_{j=0}^{p} \int_{t_j}^{t_{j+1}} e_k' F(\theta - t) b\, dt\, k_j + e_k' F(\theta) z - \alpha(z,\mu) \operatorname{sign} y_k, \quad k \in I(z)$$

$$q_l(t_j, j \in P; y) = -y' F(\theta - t_l) b, \quad l \in P$$

$$t_0 = 0, \quad t_{p+1} = \theta, \quad k_j = \mu(z) \operatorname{sign} \Delta_\mu (t_j + 0)$$

$$e_k' = (0, \dots, 0, \underset{k}{1}, 0, \dots, 0)$$

We call the system of equations (2.3) the governing equations of the OS.
Assume that the quantities z and θ are such that the relations

$$\operatorname{rank}\{e_k' F(\theta - t_j) b, \quad j = 1, 2, \dots, p; \quad k \in I(z)\} \geqslant |I(z)| - 1$$

hold.
The Jacobi matrix of the system has the form

$$G(t_i, i \in P; y) = \left\| \begin{matrix} \begin{matrix} 2e_k' F(\theta - t_i) b k_{i-1} \\ i \in P; k \in I(z) \end{matrix} & \mathbf{0} & \begin{matrix} -sign\ y_k \\ k \in I(z) \end{matrix} \\ \begin{matrix} diag(-y' F(\theta - t_i) A b \\ i \in P \end{matrix} & \begin{matrix} -e_k' F(\theta - t_i) b \\ i \in P, k \in I(z) \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} sign\ y_k, \\ k \in I(z) \end{matrix} & \mathbf{0} \end{matrix} \right\|$$

This matrix is non-degenerate under sufficiently general assumptions.

The numerical method of solving the governing equations in real time is analogous to the method [6] of solving the governing equations for the problem of synthesis. We shall indicate the necessary additions.

Suppose the stabilizer has transferred the object from the state z to the state $z + \Delta z$. To construct the set $I(z + \Delta z)$ when solving the governing equations we follow the values of $y(z)$. If $y_i(z) = 0$, $i \in I(z)$, we put $I(z + \Delta z) = I(z) \setminus \{i\}$. In addition, we monitor the behaviour of the passive output signals $x_i(\theta)$, $i \in I \setminus I(z)$. If $|x_i(\theta)| = \alpha(z, \mu)$, $i \in I \setminus I(z)$, we put $I(z + \Delta z) = I(z) \cup \{i\}$.

To calculate $\mu(z + \Delta z)$ we use the parameters ε_1 and ε_2 $(0 < \varepsilon_1 < \varepsilon_2 < \varepsilon)$

$$\mu(z + \Delta z) = \begin{cases} \mu(z), \varepsilon_1 < \alpha(z, \mu(z)) < \varepsilon_2 \\ \mu(z) - \delta(\mu(z)), \alpha(z, \mu(z)) < \varepsilon_1 \\ \mu(z) + \delta(\mu(z)), \alpha(z, \mu(z)) > \varepsilon_2 \end{cases}$$

By the choice of $\delta(\mu(z)) > 0$ we satisfy the inequality

$$\varepsilon_1 < \alpha(z+\Delta z, \mu(z+\Delta z)) < \varepsilon_2$$

3. THE ALGORITHM OF THE OPERATION OF THE OPTIMAL STABILIZER

At the initial instant $\tau = 0$ the OS generates the control

$$u^*(t) = u^0(t \mid x_0), \quad t \in T_0$$
$$|u^0(t \mid x_0)| = \mu(x_0), \quad t \in T_0$$

where $u^0(t \mid x_0)$ is the optimal programmed control of problem (1.2)–(1.5) calculated before connecting the OS. This control is found by the finite methods of [3]. It is applied to the input of the system (1.2) during the time interval $T_0 = [0, \theta]$.

Let τ be the current time and $k(\tau)$ be the number of the segment T_k which contains this instant: $k(\tau)\theta < k(\tau+1)\theta$, $\tau \in T_{k(\tau)}$. We will denote the state in which the stabilized system existed at the instant τ under the action of the control $u^*(t)$, $t \in [0, \tau]$ generated by the OS and of the realized perturbation $w^*(t)$, $t \in [0, \tau[$ by $x^*(\tau)$.

We will specify the rule by which the OS chooses the controls as follows:

$$u^*(\tau) = u^0(\tau - k(\tau)\theta \mid x^*(t_{k(\tau)})), \quad \tau \geqslant 0$$

Here ($u^0(t \mid x^*(t_{k(\tau)}))$, $t \in T_0$ is the optimal programmed control constructed as a result of the numerical solution of the governing equation (2.3) in real time.

When operating in this way the OS will generate piecewise-continuous control in real time in each specific mode of operation of the system.

4. MODIFICATIONS

One of the merits of the OS described in Sections 2 and 3 is the fact that it can operate when there are large deviations of systems from the position of equilibrium. But the magnitudes of the controls may turn out to be large. If the deviations are small it is possible to use OSs which generate controls satisfying the given restrictions. The natural problem of optimal control, for which such a damper can be constructed, has the form

$$\int_0^\theta |u(t)|\, dt \to \min, \quad \dot{x} = Ax + bu, \quad x(0) = z$$
$$|x_i(\theta)| \leqslant \varepsilon, \quad i = 1, 2, \ldots, n; \quad |u(t)| \leqslant 1, \quad t \in [0, \theta]$$

The necessary modifications which must be introduced into the constructions of Sections 1–3 may be obtained as in [6].

The second possible modification of the OS described in Section 3 involves ensuring the property of complete damping of the system $x(t) \equiv 0$ a finite time after the action of the perturbations has ceased.

This can be obtained as follows. We specify the bound u^* for admissible values of the control. Once the action of the perturbations has terminated after a finite time at the output of the optimal damper (Section 2) the control $u^*(\tau^*)$, satisfying the equality $u^*(\tau^*) = u^*$, will appear. Beginning at the instant τ^*, when solving the governing equations, we fix the number $\mu(\tau^*)$ but make the parameter θ variable, finding its current magnitude $\theta(\tau)$ from the governing equations (2.3).

5. ROBUSTNESS

One of the requirements imposed on stabilizers is their ability to function when variations of the parameters of the system being stabilized are not monitored. In other words, the stabilizer, calculated using the model $\dot{x} = Ax$, may deal with the model

$$\dot{x} = \bar{A}x \tag{5.1}$$

under practical conditions, where the matrix $\bar{A}$ differs in some sense only slightly from the matrix A. Leaving aside problems of adaptive control and the problems of identifying the matrix $\bar{A}$, let us examine the possibility of applying the above results to the problem of robust stabilization.

To do this we write the equation of real motion

$$\dot{x} = \bar{A}x + bu + \dot{w}(t)$$

in the form

$$\dot{x} = \bar{A}x + bu + w_1(t)$$

and we consider the function

$$w_1(t) = w(t) + (\bar{A} - A)x(t), \quad t \geqslant 0$$

to be the new perturbation. The dynamical system (1.8) examined above is obtained.

The results of the operation of the OS constructed in Section 3 are shown in Fig. 1 for the system

$$\ddot{x} + x = u, \quad x(0) = 2, \quad \dot{x}(0) = 1.6, \quad \theta = 4 \tag{5.2}$$

when, in practice, the equations of motion of the object with the altered parameters has the form $\ddot{x} + 1.1 = u$ (the solid curve) and $\ddot{x} + 0.9 = u$ (the dashed curve).

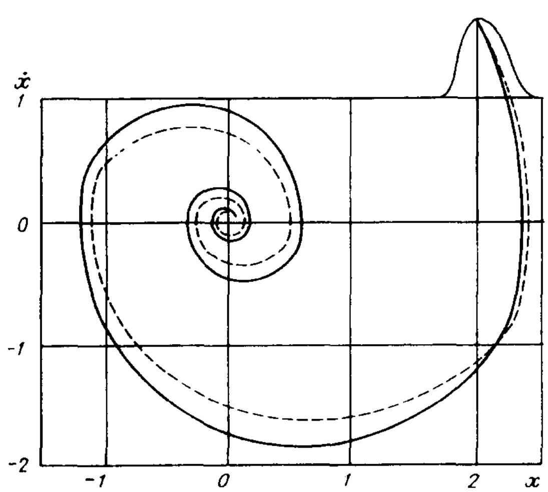

FIG. 1.

6. EXAMPLES

We will illustrate the operation of the OSs constructed in the problem of the stabilization of a linearized mathematical pendulum at the lower stable and upper unstable positions of equilibrium. When stabilizing the stable position of equilibrium the mathematical model of the dynamical system has the form (5.2).

The following values of the parameters were taken: $x(0)=2$, $\dot{x}(0)=1.6$, $\theta=4$, $w^*(t)=0.5\sin 2t$.

The states $(x_1(t)=x(t)$ and $x_2(t)=\dot{x}(t))$ of the system closed by feedback, described in Section 3 (programmed-positional control), are given below (rows (a))

	t	0	0.8	1.6	2.4	3.2	4.0	4.8
(a)	x_1	2.0	2.325	1.798	0.851	−0.167	−0.453	−0.304
	x_2	1.6	−0.395	−0.926	−1.379	−0.952	−0.070	0.335
(b)	x_1	2.0	2.285	1.404	0.093	−1.132	−1.265	−0.468
	x_2	1.6	−0.794	−1.400	−1.777	−1.033	0.634	1.182

Since, according to the algorithm of Sections 2 and 3, when this feedback is constructed, data for the feedback $u^0(t\,|\,x^*(t))$ are calculated continuously, the states of the system when using positional control closed by the latter feedback are also given (rows (b)). These data show that feedback of the second type provides a sufficiently monotone decrease of the distance from the current state of the system to the state of equilibrium. Because of this, only positional control was used in the following tests.

Graphs of the change in the optimal intensity $\mu(t)$, $t \geqslant 0$, during the stabilization are shown in Fig. 2 for the cases when there are no perturbations (the solid curve) and for the cases with the specified perturbation (the dashed curve).

The corresponding phase trajectories are shown in Fig. 3.

When stabilizing the upper unstable position of equilibrium of the pendulum the mathematical model of the system takes the form [1] $\ddot{x}-x=u$.

The following magnitudes of the parameters were chosen when the constructed OS was used: $x(0)=1.2$, $\dot{x}(0)=1$, $\theta=2$, $w^*(t)=0.5\sin 2t$. The phase trajectories are shown with the above notation in Fig. 4.

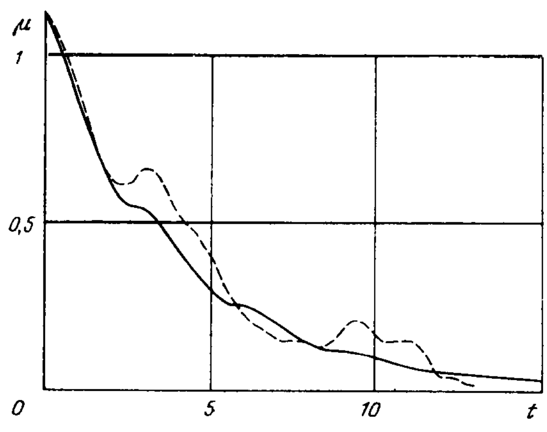

FIG. 2.

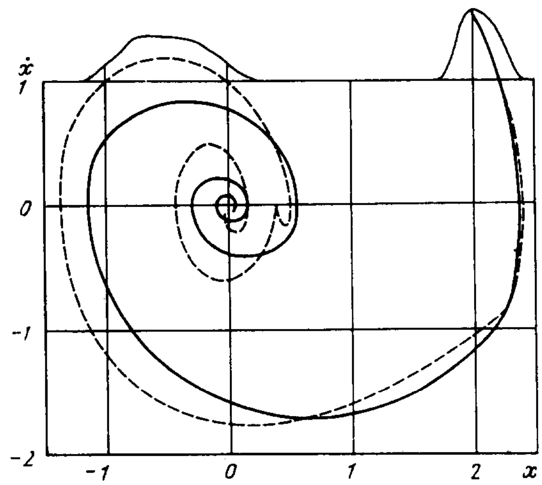

FIG. 3.

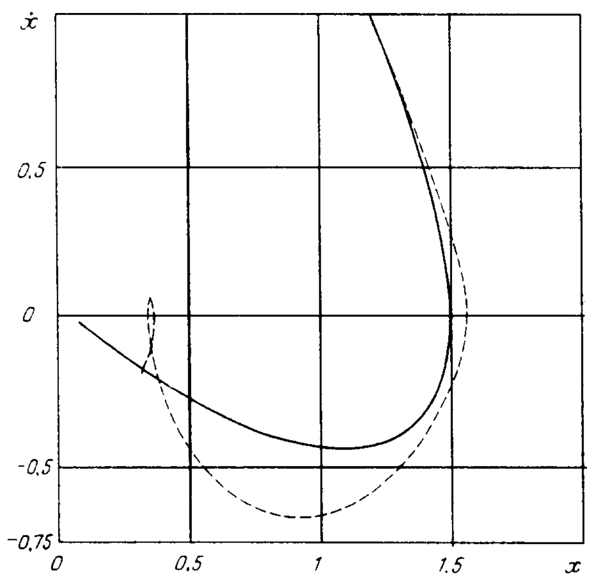

FIG. 4.

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